Operations with Concentration Inequalities

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$\alpha \boxplus \beta$? $\alpha \boxtimes \beta$? $\mathbb{P}(X + Y \geq t) \leq$?

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$\mathbb{P}(|f(X) - \mathbb{E}[f(X)]| \geq t) \leq \alpha(t)$, $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, Talagrand result

Concentration of Φ where $\|\Phi(Z) = \Phi(Z')\| \leq \max(\Lambda(Z), \Lambda(Z'))\|Z - Z'\|$.

IV - Application to Hanson-Wright inequality

Large tail concentration, Random matrix hypothesis ?

V - Concentration of bounded k^{th}-differential transformations.

I - **Motivation:** Hanson Wright for Random Matrix Theory

Theorem: **(Hanson Wright)** Given $A \in \mathcal{M}_n$ deterministic, $Z = (z_1, \dots, z_n) \in \mathbb{R}^n$ such that:

- $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz:
  $\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq C' e^{-c' t^2}$
- $\|\mathbb{E}[Z]\| \leq K$

$C, c, C', c', K > 0$, independent with n

$$\mathbb{P}\left(\left|Z^T A Z - \mathbb{E}[Z^T A Z]\right| \geq t\right) \leq C e^{-\frac{ct^2}{\|A\|_F^2}} + C e^{-\frac{ct}{\|A\|}}$$

$\Phi(Z)$ satisfying: $|\Phi(Z) - \Phi(Z')| \leq \underbrace{(\|AZ\| + \|AZ'\|)}_{\Lambda(Z)\text{:variations of } \Phi} \|Z - Z'\|$

Adamczak, Radosław (2014) *A note on the Hanson-Wright inequality for random vectors with dependencies.* Electronic Communications in Probability. 20. 10.1214/ECP.v20-3829.

II - Parallel Sum and Product.

Definition: $\alpha \boxplus \beta = (\alpha^{-1} + \beta^{-1})^{-1}$

Proposition: Given $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$, two random variables $X, Y \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$:

$$\mathbb{P}(X \geq t) \leq \alpha(t) \quad \text{and} \quad \mathbb{P}(Y \geq t) \leq \beta(t)$$

Then $\mathbb{P}(X + Y \geq t) \leq 2\alpha \boxplus \beta(t)$

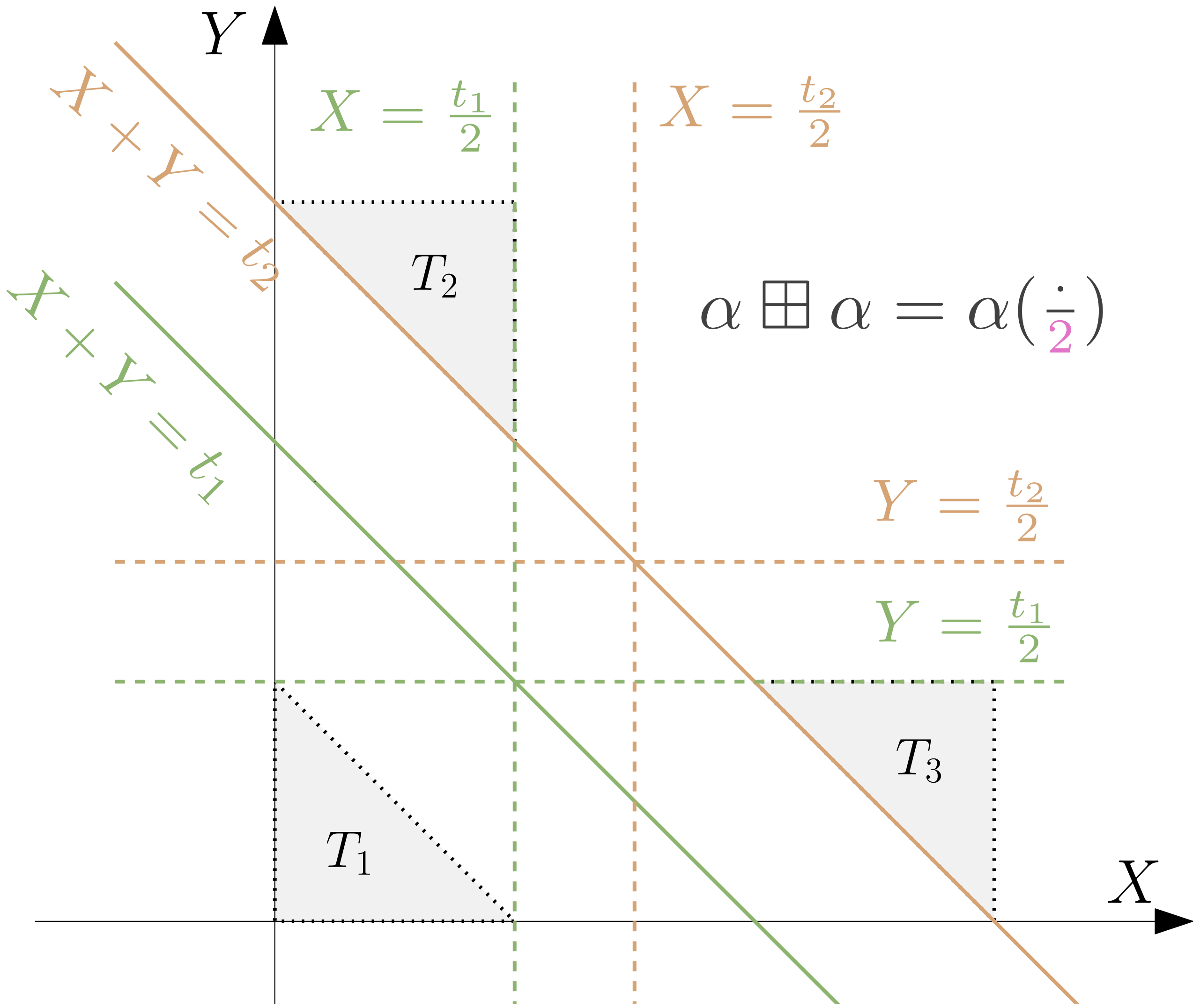
Proof: Denoting $\gamma \equiv \alpha \boxplus \beta$, for any $t \in \mathbb{R}$:

In particular: $\alpha^{-1}(\gamma(t)) + \beta^{-1}(\gamma(t)) = t$

$$\begin{aligned}\mathbb{P}(X + Y \geq t) &\leq \mathbb{P}\left(X + Y \geq \alpha^{-1}(\gamma(t)) + \beta^{-1}(\gamma(t))\right) \\ &\leq \mathbb{P}\left(X \geq \alpha^{-1}(\gamma(t))\right) + \mathbb{P}\left(Y \geq \beta^{-1}(\gamma(t))\right) \\ &\leq 2\gamma(t)\end{aligned}$$

$\forall t \in [t_1, t_2]$:

$$\mathbb{P}(X + Y \geq t) = \tfrac{2}{3} = \mathbb{P}(X \geq \tfrac{t}{2}) + \mathbb{P}(Y \geq \tfrac{t}{2})$$



$\alpha \boxplus \alpha = \alpha(\tfrac{\cdot}{2})$

Uniform distribution of (X, Y) on T_1, T_2, T_3

II - Parallel Sum and **Product**.

Definition: $\alpha \boxtimes \beta \equiv (\alpha^{-1} \cdot \beta^{-1})^{-1}$

Proposition: Given $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$, $X, Y > 0$ s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(X \geq t\right) \leq \alpha(t) \quad \text{and} \quad \mathbb{P}\left(Y \geq t\right) \leq \beta(t)$$

Then $\mathbb{P}\left(X \cdot Y \geq t\right) \leq 2\alpha \boxtimes \beta(t)$

Proof: Denoting $\gamma \equiv \alpha \boxtimes \beta = (\alpha^{-1} \cdot \beta^{-1})^{-1}$, $\forall t > 0$:

$$\begin{aligned}
\mathbb{P}\left(X \cdot Y \geq t\right) &\leq \mathbb{P}\left(X \cdot Y \geq \alpha^{-1}(\gamma(t)) \cdot \beta^{-1}(\gamma(t))\right) \\
&\leq \mathbb{P}\left(X \geq \alpha^{-1}(\gamma(t))\right) + \mathbb{P}\left(Y \geq \beta^{-1}(\gamma(t))\right) \\
&\leq 2\gamma(t)
\end{aligned}$$

II - Parallel Sum and Product.

Hanson-Wright in dimension 1

Given $X \in \mathbb{R}$ with a median m s.t. :
$\mathbb{P}(|X - m| \geq t) \leq \alpha(t)$
$\longrightarrow$ Concentration of XaX ??

$$|aX^2 - aX'^2| \leq |X - X'| \underbrace{|aX + aX'|}_{V}$$

Consequence: If $\mathbb{P}(V \geq t) \leq \beta(t)$:

$$\mathbb{P}(|aX^2 - aX'^2| \geq t) \leq \alpha \boxtimes \beta$$

2 Questions:

1. What is β ?
2. What is $\alpha \boxtimes \beta$?

1. What is β ?

$$\begin{aligned}\mathbb{P}(|V| \geq t) &\leq 2\mathbb{P}(|aX| \geq \tfrac{t}{2}) \\ &\leq 2\mathbb{P}(|aX - am| \geq t - |am|) \\ &\leq 4\mathbb{P}(|aX - am| + |am| \geq t) \\ &\leq 4(\alpha \circ (\tfrac{\mathrm{Id}}{|a|})) \boxplus (\alpha \circ \mathrm{inc}_{|am|}) \\ &\leq 4\alpha \circ (\tfrac{\mathrm{Id}}{|a|} \boxplus \mathrm{inc}_{|am|})\end{aligned}$$

Introduce: $\mathrm{inc}_u : \mathbb{R} \longrightarrow \bar{\mathbb{R}}$

$$t \longmapsto \begin{cases} 0 & \text{if } t \leq u \\ +\infty & \text{if } t > u, \end{cases}$$

$|am|$ constant: $\mathbb{P}(|am| \geq t) \leq \alpha \circ \mathrm{inc}_{|am|}(t)$

II - Parallel Sum and Product.

Hanson-Wright in dimension 1

Given $X \in \mathbb{R}$ with a median m s.t. :
$\mathbb{P}(|X - m| \geq t) \leq \alpha(t)$
$\longrightarrow$ Concentration of XaX ??

$$|aX^2 - aX'^2| \leq |X - X'| \underbrace{|aX + aX'|}_{V}$$

$$\mathbb{P}(|aX^2 - aX'^2| \geq t) \leq \alpha \boxtimes \beta$$

- $\alpha \circ (f \boxtimes g) = (\alpha \circ f) \boxtimes (\alpha \circ g)$
- $f \boxtimes (g \boxplus h) = (f \boxtimes g) \boxplus (f \boxtimes h)$
- $\mathrm{inc}_u^{-1} : t \mapsto u$
- $\mathrm{Id} \boxtimes \mathrm{inc}_u = (u\,\mathrm{Id})^{-1} = \frac{\mathrm{Id}}{u}$
- $\min(f, g) \circ \frac{\mathrm{Id}}{2} \leq f \boxplus g \leq \min(f, g)$

1. What is β ?

$$\mathbb{P}(|V| \geq t) \leq 4\alpha \circ (\tfrac{\mathrm{Id}}{|a|} \boxplus \mathrm{inc}_{|am|})$$

2. What is $\alpha \boxtimes \beta$?

$$\begin{aligned} \alpha \boxtimes \beta &\leq 4\alpha \boxtimes \alpha \circ (\tfrac{\mathrm{Id}}{|a|} \boxplus \mathrm{inc}_{|am|}) \\ &\leq 4\alpha \circ \left(\mathrm{Id} \boxtimes (\tfrac{\mathrm{Id}}{|a|} \boxplus \mathrm{inc}_{|am|})\right) \\ &\leq 4\alpha \circ \left((\mathrm{Id} \boxtimes \tfrac{\mathrm{Id}}{|a|}) \boxplus (\mathrm{Id} \boxtimes \mathrm{inc}_{|am|})\right) \\ &\leq 4\alpha \circ \left(\sqrt{\tfrac{\mathrm{Id}}{|a|}} \boxplus \tfrac{\mathrm{Id}}{|am|}\right) \\ &\leq 4\alpha \circ \min\left(\sqrt{\tfrac{\mathrm{Id}}{2|a|}}, \tfrac{\mathrm{Id}}{2|am|}\right) \end{aligned}$$

Hanson Wright with:

- $|am| = \|A\|_F$
- $a = \|A\|$

If $\alpha : t \mapsto e^{-t^2}$:

$$\leq 4e^{-\frac{t^2}{4|am|^2}} + 4e^{-\frac{t}{2a}}$$

III - Concentration in High Dimension

Theorem: Given $Z \sim \mathcal{N}(\mu, I_n)$, $\forall f : \mathbb{R}^n \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq 2e^{-\frac{t^2}{2}} \qquad Z, Z' \ i.i.d.$$

Given $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ λ-Lipschitz and $f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz:

$$\mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z'))| \geq t\right)$$

$$= \mathbb{P}\left(\left|\frac{1}{\lambda}f(\Phi(Z)) - \frac{1}{\lambda}f(\Phi(Z'))\right| \geq \frac{t}{\lambda}\right) \leq 2e^{-\frac{t^2}{2\lambda^2}}.$$

$$\|\Phi(Z) - \Phi(Z')\| \leq \Lambda \|Z - Z'\| \quad a.s.$$

Random

Theorem: (Talagrand)
Given $Z = (Z_1, \dots Z_n) \in [0,1]^n$ s.t. $Z_1, \dots, Z_n$ independent
$\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz and convex:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq 2e^{-\frac{t^2}{4}}.$$

Michel Talagrand (1995) *Concentration of measure and isoperimetric inequalities in product spaces.* Publications mathématiques de l'IHÉS, 104:905–909.

III - Concentration in High Dimension

Theorem: Consider $Z \in \mathbb{R}^n$, random, s.t. $\forall f : \mathbb{R}^n \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t) \qquad (Z, Z' \text{ i.i.d.})$$

- Consider $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(\Lambda(Z) \geq t\right) \leq \beta(t)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq \max(\Lambda(Z), \Lambda(Z'))\|Z - Z'\| \quad a.s.$$

Then: $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0 : \quad \mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) \leq 2\, \alpha \boxtimes \beta(t)$$

Proof: Denote $\Lambda = \Lambda(Z)$, $\Lambda' = \Lambda(Z')$, $\gamma \equiv \alpha \boxtimes \beta = (\alpha^{-1} \cdot \beta^{-1})^{-1}$, $\theta \equiv \beta^{-1}(\gamma(t))$

$$\mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) \leq \underbrace{\mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t, \max(\Lambda, \Lambda') \leq \theta\right)}_{\leq \mathbb{P}(|h(\Phi(Z)) - h(\Phi(Z')| \geq t) \;\; \leq \alpha\left(\frac{t}{\beta^{-1}(\gamma(t))}\right)} + \underbrace{\mathbb{P}\left(\max(\Lambda, \Lambda') \geq \theta\right)}_{\leq \beta(\beta^{-1}(\gamma(t))}$$

With $h : x \mapsto \sup_{\Lambda(z) \leq \theta} f \circ \Phi(z) - \theta d(x, z)$

$\to$ equal to $f \circ \phi$ on $\{z \in \mathbb{R}^n, \Lambda(z) \leq \theta\}$.

$\to$ $\beta^{-1}(\gamma(t))$-Lipschitz on $\mathbb{R}^n$

$$\leq \alpha(\alpha^{-1}(\gamma(t)))$$

(Since $\forall t > 0 : \; \alpha^{-1}(\gamma(t)) \cdot \beta^{-1}(\gamma(t)) = t$)

III - Concentration in High Dimension

Theorem: Consider $Z \in \mathbb{R}^n$, random, s.t. $\forall f : \mathbb{R}^n \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t) \qquad (Z, Z' \text{ i.i.d.})$$

- Consider $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(\Lambda(Z) \geq t\right) \leq \beta(t)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq \max(\Lambda(Z), \Lambda(Z'))\|Z - Z'\| \quad a.s.$$

Then: $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0 : \quad \mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) \leq 2\, \alpha \boxtimes \beta(t)$$

Theorem: Consider $X = (X_1, \dots, X_n) \in \mathbb{R}^n$,
Such that $\forall i \in [n]$:
- $X_i = \phi(Z_i)$
- $Z_i \sim \mathcal{N}(0,1), i = 1, \dots, n \quad idpts$
- $\log \circ \phi' \circ \sqrt{\cdot}|_{[\log 4, \infty]}$ subaditive

Then:

$$\mathbb{P}(|f(X) - f(X')|| \geq t) \leq \exp\left(-(\mathrm{Id} \cdot \phi')^{-1} \left(\frac{t}{\phi'(2\sqrt{\log(2n)})} \right)^2 /2 \right)$$

"Large-tailed concentration inequality"

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^n \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t) \qquad (Z, Z' \text{ i.i.d.})$$

- Consider $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(\Lambda(Z) \geq t\right) \leq \beta(t)$$

Then: $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0 : \quad \mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) \leq 2\, \alpha \boxtimes \beta(t)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq \max(\Lambda(Z), \Lambda(Z'))\|Z - Z'\| \quad a.s.$$

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq 2e^{-t^2/2}$$

$$\Longrightarrow \mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq Ce^{-ct^2}$$

$$\Longrightarrow \mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq C'e^{-c't}$$

For $C, C', c', c > 0$ numerical constant.

Yes, **IF** $\alpha, \beta : t \mapsto 2e^{-t^2/2}$

Other choices for α, β ??

Question: Possible to replace $\begin{cases} f(Z') \\ f(\Phi(Z')) \end{cases}$ with $\begin{cases} \mathbb{E}[f(Z)] \\ \mathbb{E}[f(\Phi(Z))] \end{cases}$??

IV - Application to Hanson-Wright inequality

Convex concentration setting (Talagrand's Theorem)

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^n \to \mathbb{R}$, 1-Lipschitz, convex:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t)$$

- Consider $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(|\Lambda(Z) - \mathbb{E}[\Lambda(Z)]| \geq t\right) \leq \alpha\left(\frac{t}{\lambda}\right)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ convex s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq \max(\Lambda(Z), \Lambda(Z'))\|Z - Z'\| \quad a.s.$$

- Assume α independent with n and:

$$\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty \quad (\mathbb{E}[|f(Z) - \mathbb{E}[f(Z)]|^2] \leq \sigma_\alpha^2)$$

Then:

$$\forall t > 0 : \quad \mathbb{P}\left(|\Phi(Z) - \mathbb{E}[\Phi(Z)]| \geq t\right) \leq 2\,\alpha\left(\frac{t}{|\mathbb{E}[\Lambda(Z)]|}\right) + 2\,\alpha\left(\sqrt{\frac{t}{\lambda}}\right).$$

Recall:

$$\alpha \boxtimes \alpha \circ \min\left(\mathrm{inc}_{\mathbb{E}[\Lambda(Z)]}, \frac{\mathrm{Id}}{\lambda}\right) = \alpha \circ \min\left(\frac{\mathrm{Id}}{\mathbb{E}[\Lambda(Z)]}, \sqrt{\frac{\mathrm{Id}}{\lambda}}\right)$$

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz, convex:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t)$$

- Consider $\Lambda : \mathbb{R}^n \to \mathbb{R}_+$ s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(|\|AZ\| - \mathbb{E}[\|AZ\|]| \geq t\right) \leq \alpha\left(\frac{t}{\|A\|}\right)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ convex s.t.:

$$\|Z^T AZ - Z'^T AZ'\| \leq 2\max(\underbrace{\|AZ\|, \|AZ'\|}_{\Lambda(Z)})\|Z - Z'\| \quad a.s$$

- Assume α independent with n and:

$$\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty$$

Then: $\forall A \in \mathcal{M}_n$:

$$\forall t \geq 0 : \quad \mathbb{P}\left(\left|Z^T AZ - \mathbb{E}[Z^T AZ]\right| \geq t\right) \leq 2\,\alpha\left(\frac{t}{\mathbb{E}[\|AZ\|]}\right) + 2\,\alpha\left(\sqrt{\frac{t}{\|A\|}}\right).$$

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz, convex:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t) \qquad (*)$$

with $\alpha : \mathbb{R}^+ \to \mathbb{R}_+$ independent with n.

- $\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty$

- Assume $\|\mathbb{E}[Z]\| \leq \sigma_\alpha$.

Then: $\forall A \in \mathcal{M}_n$, $\forall t > 0$:

$$\mathbb{P}\left(\left|Z^T AZ - \mathbb{E}[Z^T AZ]\right| \geq t\right) \leq C\ \alpha\left(\frac{ct}{\sigma_\alpha \|A\|_F}\right) + C\alpha\left(\sqrt{\frac{ct}{\|A\|}}\right).$$

$$\mathbb{E}[\|AZ\|] \leq \sqrt{\mathbb{E}[\|AZ\|^2]}$$
$$= \sqrt{\mathbb{E}[\mathrm{Tr}(A^T AZZ^T)]}$$
$$= \|A\|_F \sqrt{\|\mathbb{E}[ZZ^T]\|}$$

Lemma: Given $Z \in \mathbb{R}^n$ satisfying $(*)$:

$$\|\mathbb{E}[ZZ^T]\| \leq \|\mathbb{E}[Z]\|^2 + C\sigma_\alpha^2$$

for some numerical constant $C > 0$

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz, convex:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t) \qquad (*)$$

with $\alpha : \mathbb{R}^+ \to \mathbb{R}_+$ independent with n.

- $\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty$
- Assume $\|\mathbb{E}[Z]\| \leq \sigma_\alpha$.

Then: $\forall A \in \mathcal{M}_n$, $\forall t > 0$:

$$\mathbb{P}\left(\left|Z^T AZ - \mathbb{E}[Z^T AZ]\right| \geq t\right) \leq C\ \alpha\left(\frac{ct}{\sigma_\alpha \|A\|_F}\right) + C\alpha\left(\sqrt{\frac{ct}{\|A\|}}\right).$$

Comparison *Adamczak's result:* $\alpha : t \mapsto e^{-\frac{t^2}{2\sigma_\alpha{}^2}}$

V - Concentration of bounded k^{th}-differential transformations.

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz, convex:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t)$$

with $\alpha : \mathbb{R}^+ \to \mathbb{R}_+$.

Then, Given $d \in \mathbb{N}$, $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^p$ d-times differentiable:

$$\mathbb{P}\left(|\Phi(Z) - m_0| \geq t\right) \leq C_d\ \alpha \circ \beta_0(c_d\ t),$$

where, $\forall k \in [d-1]$, we introduced m_k, a median of $\|d^k\Phi|_Z\|$ and $m_d \equiv \sup_{z\in\mathbb{R}^n} \|d^d\Phi|_z\|$.

Proof: Denote: $\beta_k \equiv \left(\frac{\mathrm{Id}}{m_{k+1}}\right)^{\frac{1}{1}} \boxplus \cdots \boxplus \left(\frac{(d-k)!\,\mathrm{Id}}{m_d}\right)^{\frac{1}{d-k}}$

Strategy: Show recursively for $k = d-1, \dots, 0$:

$$\mathbb{P}\left(\left|\|d^k\Phi|_Z\| - m_k\right| \geq t\right) \leq C\ \alpha\left(c\ \beta_k(t)\right),$$

Last step $k = 0$*:* Given $t \geq 0$, denote:

$$\mathcal{A}_t \equiv \left\{z \in \mathbb{R}^n : \forall l \in [d], \left|\|d^l\Phi|_z\| - m_l\right| \leq \beta_l^{-1}\left(\beta_0(t)\right)\right\}.$$

Core inference: Φ is ω_t-continuous on $\mathcal{A}_t$ with certain $\omega_t : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\omega_t^{-1}(t) \geq c\beta_0(t)$

- $\mathbb{P}\left(|\Phi(Z) - m_0| \geq t,\ Z \in \mathcal{A}_t\right) \leq 2\alpha \circ \omega_t^{-1}(t) \leq C\alpha(c\beta_0\,(t))$
- $\mathbb{P}\left(Z \notin \mathcal{A}_t\right) \leq \sum_{l=1}^{d} \mathbb{P}\left(\left|\|d^l\Phi|_z\| - m_l\right| \geq \beta_l^{-1}\left(\beta_k(t)\right)\right) \leq \sum_{l=1}^{d} C\ \alpha\left(c\beta_l \circ \beta_l^{-1} \circ \beta_0(t)\right) \leq C'\alpha(c\beta_0(t))$