Operations with Concentration Inequalities

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SCHOOL OF DATA SCIENCE

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Given two random variables: $X, Y \in \mathbb{R}$

$$\forall t > 0 : \quad \mathbb{P}\left(X \geq t\right) \leq \alpha(t)$$

$$\times$$

$$\forall t > 0 : \quad \mathbb{P}\left(Y \geq t\right) \leq \beta(t)$$

$$=$$

$$\forall t > 0 :$$

$$\mathbb{P}\left(X \cdot Y \geq t\right) \leq 2\, \alpha \boxtimes \beta(t)$$

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Given two random variables: $X, Y \in \mathbb{R}$

$$\forall t > 0 : \quad \mathbb{P}(X \geq t) \leq \alpha(t)$$

$$\times$$

$$\forall t > 0 : \quad \mathbb{P}(Y \geq t) \leq \beta(t)$$

$$=$$

"Parallel product"

$$\forall t > 0 : \quad \mathbb{P}(X \cdot Y \geq t) \leq 2\, \alpha \boxtimes \beta(t)$$

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- Given a random vector : $X \in \mathbb{R}^n$:
  $\forall t > 0 : \quad \mathbb{P}\left(|f(X) - \mathbb{E}[f(X)]| \geq t\right) \leq \alpha(t)$
  $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz

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Content

I - **Motivation:** Hanson Wright for Random Matrix Theory

II - Parallel Sum and Product.

$\alpha \boxplus \beta$? $\alpha \boxtimes \beta$? $\mathbb{P}(X + Y \geq t) \leq$?

III - Concentration in High Dimension

$\mathbb{P}(|f(X) - \mathbb{E}[f(X)]| \geq t) \leq \alpha(t)$, $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, Talagrand result
Concentration of Φ where $\|\Phi(Z) = \Phi(Z')\| \leq V\|Z - Z'\|$.

IV - Application to Hanson-Wright inequality

Large tail concentration, Random matrix hypothesis ?

I - **Motivation:** Hanson Wright for Random Matrix Theory

Given $x_1, \dots, x_n \sim \mathcal{N}(0, \Sigma)$, i.i.d. random vectors, note $X \equiv (x_1, \dots, x_n) \in \mathbb{R}^{n \times p}$.

Goal: Eigen value distribution of $\frac{1}{n} X X^T$: $\mu \equiv \frac{1}{p} \sum_{i=1}^p \delta_{\lambda_i}$ **??**

Eigen values of $\frac{1}{p} X X^T$

$$\left(\text{Sp} \left(\frac{1}{p} X X^T \right) = \{ \lambda_1, \dots, \lambda_p \} \right)$$

- Correspondance $\mu \longleftrightarrow m : z \mapsto \int_{\mathbb{R}} \frac{1}{z - \lambda} d\mu(\lambda)$

"Steiltjes Transform" (similar to Cauchy Transform)

- Link with the *"Resolvent"*: $m(z) = \frac{1}{p} \text{Tr} Q(z)$, where $Q(z) \equiv \left(z I_p - \frac{1}{n} X X^T \right)^{-1}$.

Strategy: Find deterministic $\tilde{Q} \in \mathcal{M}_p$ such that $Q \approx \tilde{Q}$

I - **Motivation:** Hanson Wright for Random Matrix Theory

Goal: Approach $\mathbb{E}[Q] = \mathbb{E}\left[\left(zI_p - \frac{1}{n}XX^T\right)^{-1}\right]$

- Of course $\mathbb{E}[Q]$ far from $(zI_p - \Sigma)^{-1}$

$\Sigma \equiv \mathbb{E}\left[\frac{1}{n}XX^T\right] = \mathbb{E}[x_i x_i^T], \ \forall i \in [n]$

Solution: Look for $\tilde{Q} \equiv \left(zI_p - \frac{\Sigma}{1+\delta}\right)^{-1}$

δ to be determined

Given $A \in \mathcal{M}_p$, deterministic:

Dependence between Q and x_i

$$\mathrm{Tr}\left(A(\mathbb{E}[Q] - \tilde{Q})\right) = \mathbb{E}\left[\mathrm{Tr}\left(AQ\left(\frac{\Sigma}{1+\delta} - \frac{1}{n}XX^T\right)\tilde{Q}\right)\right] = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\left[\mathrm{Tr}\left(\frac{AQ\Sigma\tilde{Q}}{1+\delta} - AQx_ix_i^T\tilde{Q}\right)\right]$$

I - **Motivation:** Hanson Wright for Random Matrix Theory

$$\mathrm{Tr}\left(A(\mathbb{E}[Q] - \tilde{Q}_\delta)\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Tr}\left(\left(\frac{1}{1+\delta} - \frac{1}{1+\frac{1}{n}x_i^T Q_{-i} x_i}\right) A Q_{-i} x_i x_i^T \tilde{Q}_\delta\right)\right] + O\left(\frac{1}{\sqrt{n}}\right)$$

Use the *Schur Formula:* $Qx_i = \frac{Q_{-i}x_i}{1+\frac{1}{n}x_i^T Q_{-i} x_i}$, with $Q_{-i} \equiv \left(zI_p - \frac{1}{n}XX^T - x_i x_i^T\right)^{-1}$.

Independent with x_i

independent with p, n.

1. Chose $\delta_1 \equiv \frac{1}{n}\mathbb{E}[x_i^T Q_{-i} x_i] \approx \frac{1}{n}\mathrm{Tr}(\Sigma\mathbb{E}[Q]) \approx \frac{1}{n}\mathrm{Tr}(\Sigma\tilde{Q}_{\delta_1})$

   Hanson-Wright Inequality: $\mathbb{P}\left(\left|\frac{1}{n}x_i^T Q_{-i} x_i - \delta_1\right| \geq t\right) \leq Ce^{-ct^2}$

2. Chose δ_2 solution to $\delta = \frac{1}{n}\mathrm{Tr}(\Sigma\tilde{Q}_\delta)$ ⟶ $\mathrm{Tr}\left(A(\mathbb{E}[Q] - \tilde{Q}_{\delta_2})\right) = O\left(\frac{1}{\sqrt{n}}\right)$

Theorem: (Hanson Wright) Given $A \in \mathcal{M}_n$ deterministic, $Z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ such that:

- $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz:
  $$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq C' e^{-c' t^2}$$
- $\|\mathbb{E}[Z]\| \leq K$

$C, c, C', c', K > 0$, independent with n

$$\mathbb{P}\left(\left|Z^T A Z - \mathbb{E}[Z^T A Z]\right| \geq t\right) \leq C e^{-\frac{ct^2}{\|A\|_F^2}} + C e^{-\frac{ct}{\|A\|}}$$

$\Phi(Z)$ satisfying: $|\Phi(Z) - \Phi(Z')| \leq \underbrace{(\|AZ\| + \|AZ'\|)}_{V\text{:variations of }\Phi} \|Z - Z'\|$

Adamczak, Radosław (2014) *A note on the Hanson-Wright inequality for random vectors with dependencies.* Electronic Communications in Probability. 20. 10.1214/ECP.v20-3829.

II - Parallel Sum and Product.

Definition: $\alpha \boxplus \beta = (\alpha^{-1} + \beta^{-1})^{-1}$

Proposition: Given $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$, two random variables $X, Y \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$:

$$\mathbb{P}(X \geq t) \leq \alpha(t) \quad \text{and} \quad \mathbb{P}(Y \geq t) \leq \beta(t)$$

Then $\mathbb{P}(X + Y \geq t) \leq 2\alpha \boxplus \beta(t)$

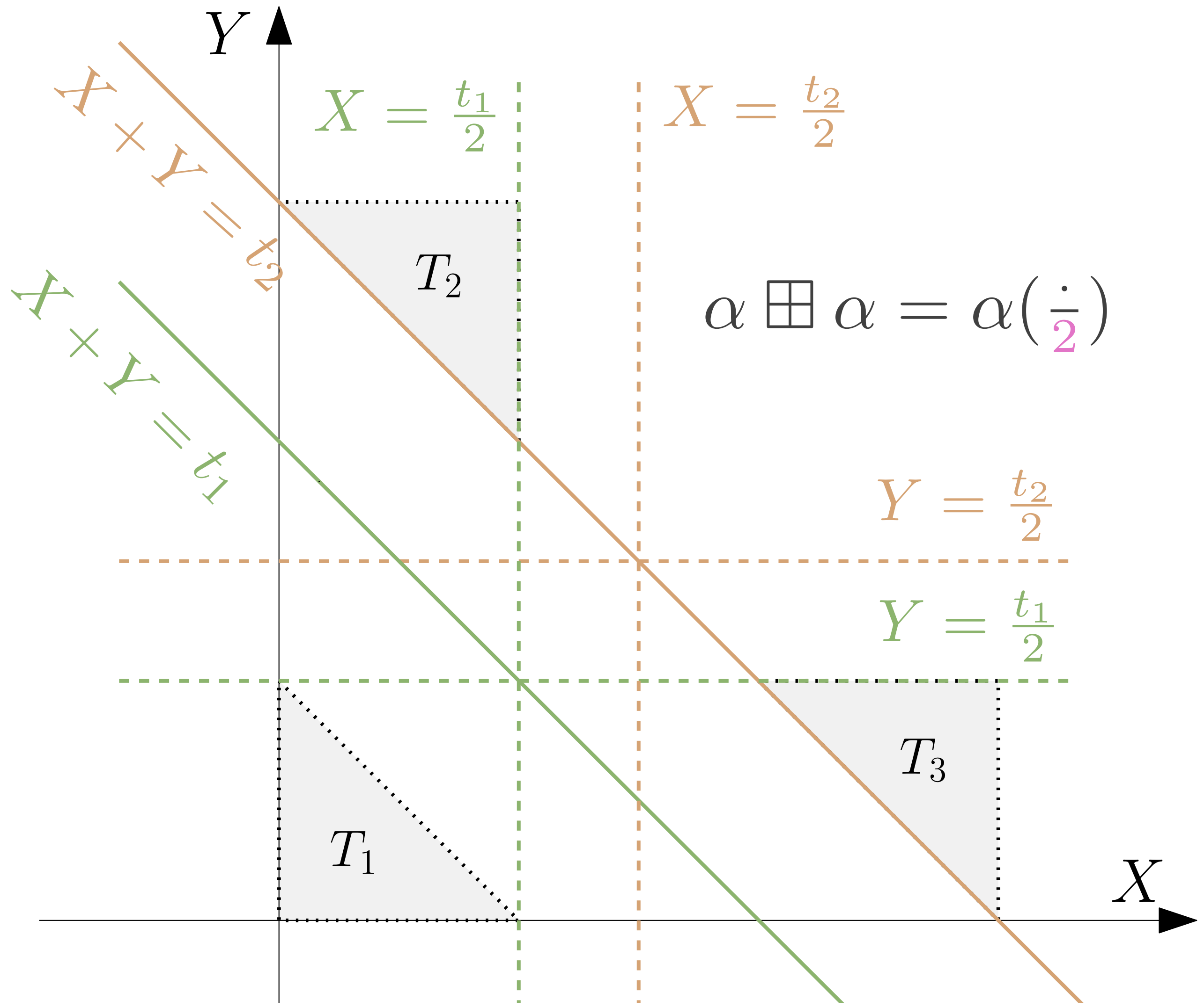
Proof: Denoting $\gamma \equiv \alpha \boxplus \beta$, for any $t \in \mathbb{R}$:

In particular: $\alpha^{-1}(\gamma(t)) + \beta^{-1}(\gamma(t)) = t$

$$\begin{aligned}\mathbb{P}(X + Y \geq t) &\leq \mathbb{P}\left(X + Y \geq \alpha^{-1}(\gamma(t)) + \beta^{-1}(\gamma(t))\right)\\ &\leq \mathbb{P}\left(X \geq \alpha^{-1}(\gamma(t))\right) + \mathbb{P}\left(Y \geq \beta^{-1}(\gamma(t))\right)\\ &\leq 2\gamma(t)\end{aligned}$$

$\forall t \in [t_1, t_2]$:

$$\mathbb{P}(X + Y \geq t) = \tfrac{2}{3} = \mathbb{P}(X \geq \tfrac{t}{2}) + \mathbb{P}(Y \geq \tfrac{t}{2})$$



Uniform distribution of (X, Y) on T_1, T_2, T_3

II - Parallel Sum and **Product**.

Definition: $\alpha \boxtimes \beta \equiv (\alpha^{-1} \cdot \beta^{-1})^{-1}$ $(\alpha, \beta > 0)$

$\alpha, \beta : (-\infty, 0) \to \{+\infty\}$

Proposition: Given $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$, $X, Y > 0$ s.t.:

$\forall t > 0 : \quad \mathbb{P}(X \geq t) \leq \alpha(t) \quad$ and $\quad \mathbb{P}(Y \geq t) \leq \beta(t)$

Then $\mathbb{P}(X \cdot Y \geq t) \leq 2\alpha \boxtimes \beta(t)$

Proof: Denoting $\gamma \equiv \alpha \boxtimes \beta = (\alpha^{-1} \cdot \beta^{-1})^{-1}$, $\forall t > 0$:

$$\mathbb{P}(X \cdot Y \geq t) \leq \mathbb{P}\left(X \cdot Y \geq \alpha^{-1}(\gamma(t)) \cdot \beta^{-1}(\gamma(t))\right)$$

$$\leq \mathbb{P}\left(X \geq \alpha^{-1}(\gamma(t))\right) + \mathbb{P}\left(Y \geq \beta^{-1}(\gamma(t)\right.$$

$$\leq 2\gamma(t)$$

II - Parallel Sum and **Product**.

Introduce:

$$\mathrm{inc}_u : \quad \mathbb{R} \longrightarrow \bar{\mathbb{R}}$$
$$t \longmapsto \begin{cases} -\infty & \text{if } t \leq u \\ +\infty & \text{if } t > u, \end{cases}$$

Lemma: Given α decreasing:

$$\mathbb{P}(|V - u| \geq t) \leq \alpha(t)$$
$$\implies \mathbb{P}(|V| \geq t) \leq \alpha \circ \min\left(\mathrm{inc}_{2u}, \frac{\mathrm{Id}}{2}\right)(t)$$

Proof: $t \geq 2u \implies \frac{t}{2} \leq t - u.$

Lemma: $\alpha \circ (f \boxplus g) = (\alpha \circ f) \boxplus (\alpha \circ g)$

- $\min(f, g)^{-1} = \max(f^{-1}, g^{-1})$
- $\mathrm{inc}_u^{-1} : t \mapsto u$

Now, consider X, V:

$$\mathbb{P}(X \geq t) \leq \alpha \qquad \mathbb{P}(|V - u| \geq t) \leq \alpha(t/\lambda),$$
$$\to \mathbb{P}(XV \geq t) \leq \alpha \circ \mathrm{Id} \boxtimes \min(\mathrm{inc}_{2u}, \mathrm{Id}/2\lambda)(t)$$

Lemma: $\mathrm{Id} \boxtimes \min\left(\mathrm{inc}_u, \frac{\mathrm{Id}}{\lambda}\right) = \min\left(\frac{\mathrm{Id}}{u}, \sqrt{\frac{\mathrm{Id}}{\lambda}}\right)$

Proof: $\mathrm{Id}^{-1} \cdot \min\left(\mathrm{inc}_u, \frac{\mathrm{Id}}{\lambda}\right)^{-1} = \mathrm{Id} \cdot \max(u, \lambda \,\mathrm{Id})$

$$= \max\left(u\,\mathrm{Id}, \lambda\,\mathrm{Id}^2\right)$$

If $\alpha : t \mapsto e^{-t^2}$:

$$\leq e^{-\frac{t^2}{u^2}} + e^{-\frac{t}{\lambda}}$$

Retrieve Hanson Wright right-hand term with:

- $u = \|A\|_F$
- $\lambda = \|A\|$

III - Concentration in High Dimension

Theorem: Given $Z \sim \mathcal{N}(\mu, I_n)$, $\forall f : \mathbb{R}^n \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq 2e^{-\frac{t^2}{2}} \quad Z, Z' \ i.i.d.$$

Given $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ λ-Lipschitz and $f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz:

$$\mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z'))| \geq t\right)$$

$$= \mathbb{P}\left(\left|\frac{1}{\lambda} f(\Phi(Z)) - \frac{1}{\lambda} f(\Phi(Z'))\right| \geq \frac{t}{\lambda}\right) \leq 2e^{-\frac{t^2}{2\lambda^2}}.$$

$$\|\Phi(Z) - \Phi(Z')\| \leq V \|Z - Z'\| \quad a.s.$$

Random

Theorem: **(Talagrand)**
Given $Z = (Z_1, \dots Z_n) \in [0,1]^n$ s.t. $Z_1, \dots, Z_n$ independent
$\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz and convex:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq 2e^{-\frac{t^2}{4}}.$$

Michel Talagrand (1995) *Concentration of measure and isoperimetric inequalities in product spaces.* Publications mathématiques de l'IHÉS, 104:905–909.

III - Concentration in High Dimension

Theorem: Consider $Z \in \mathbb{R}^n$, random, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t) \qquad (Z, Z' \text{ i.i.d.})$$

- Consider $V \in \mathbb{R}_+$ random s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(V \geq t\right) \leq \beta(t)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq V \|Z - Z'\| \quad a.s.$$

Then: $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0 : \quad \mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) \leq 2\, \alpha \boxtimes \beta(t)$$

Proof: Denote $\gamma \equiv \alpha \boxtimes \beta = (\alpha^{-1} \cdot \beta^{-1})^{-1}$ In particular, $\forall t > 0 : \ \alpha^{-1}(\gamma(t)) \cdot \beta^{-1}(\gamma(t)) = t$

$$\begin{aligned}
\mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) &\leq \mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t, V \leq \beta^{-1}(\gamma(t))\right) + \mathbb{P}\left(V \geq \beta^{-1}(\gamma(t))\right) \\
&\leq \alpha\left(\frac{t}{\beta^{-1}(\gamma(t))}\right) + \beta(\beta^{-1}(\gamma(t))) \\
&\leq 2\gamma(t)
\end{aligned}$$

III - Concentration in High Dimension

Theorem: Consider $Z \in \mathbb{R}^n$, random, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t) \qquad (Z, Z' \text{ i.i.d.})$$

- Consider $V \in \mathbb{R}_+$ random s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(V \geq t\right) \leq \beta(t)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq V \|Z - Z'\| \quad a.s.$$

Then: $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0 : \quad \mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) \leq 2\, \alpha \boxtimes \beta(t)$$

Theorem: Consider $X = (X_1, \dots, X_n) \in \mathbb{R}^n$,
Such that $\forall i \in [n]$:
- $X_i = \phi(Z_i)$
- $Z_i \sim \mathcal{N}(0, 1)$

Then: $\mathbb{P}(|f(X) - f(X')|| \geq t)$

$$\leq \exp\left(- \min\left(\frac{t}{\phi'(2\sqrt{\log(2n)})}, \frac{\left(\mathrm{Id} \cdot \phi'\right)^{-1}(t)}{2} \right)^2 \right)$$

" Large-tailed concentration inequality "

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t) \qquad (Z, Z' \text{ i.i.d.})$$

- Consider $V \in \mathbb{R}_+$ random s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(V \geq t\right) \leq \beta(t)$$

Then: $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0 : \quad \mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z')| \geq t\right) \leq 2\, \alpha \boxtimes \beta(t)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq V\|Z - Z'\| \quad a.s.$$

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq 2e^{-t^2/2}$$

$$\Longrightarrow \mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq Ce^{-ct^2}$$

$$\Longrightarrow \mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq C'e^{-c't}$$

For $C, C', c', c > 0$ numerical constant.

Yes, **IF** $\alpha, \beta : t \mapsto 2e^{-t^2/2}$

Other choices for α, β ??

Question: Possible to replace $\begin{cases} f(Z') \\ f(\Phi(Z')) \end{cases}$ with $\begin{cases} \mathbb{E}[f(Z)] \\ \mathbb{E}[f(\Phi(Z))] \end{cases}$ **??**

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^n \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t)$$

- Consider $V \in \mathbb{R}_+$ random s.t.:

$$\forall t > 0: \quad \mathbb{P}\left(|V - \mathbb{E}[V]| \geq t\right) \leq \alpha\left(\frac{t}{\lambda}\right)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.:

$$\|\Phi(Z) - \Phi(Z')\| \leq V\|Z - Z'\| \quad a.s.$$

- Assume α independent with n and:

$$\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty \quad (\mathbb{E}[|f(Z) - \mathbb{E}[f(Z)]|^2] \leq \sigma_\alpha^2)$$

Then: $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0: \quad \mathbb{P}\left(|f(\Phi(Z)) - \mathbb{E}[f(\Phi(Z))]| \geq t\right) \leq C\,\alpha\left(\frac{t}{c|\mathbb{E}[V]|}\right) + C\,\alpha\left(\sqrt{\frac{t}{c\lambda}}\right).$$

Recall:

$$\alpha \boxtimes \alpha \circ \min\left(\mathrm{inc}_{\mathbb{E}[V]}, \frac{\mathrm{Id}}{\lambda}\right) = \alpha \circ \min\left(\frac{\mathrm{Id}}{\mathbb{E}[V]}, \sqrt{\frac{\mathrm{Id}}{\lambda}}\right)$$

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t)$$

- Consider $V \in \mathbb{R}_+$ random s.t.:

$$\forall t > 0 : \quad \mathbb{P}\left(|\|AZ\| - \mathbb{E}[\|AZ\|]| \geq t\right) \leq \alpha\left(\frac{t}{\|A\|}\right)$$

- Consider $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ s.t.:

$$\left\|Z^T AZ - Z'^T AZ'\right\| \leq \underbrace{\left(\|AZ\| + \|AZ'\|\right)}_{V} \|Z - Z'\| \quad a.s.$$

- Assume α independent with n and:

$$\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty$$

Then: $\forall A \in \mathcal{M}_n$, $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\forall t > 0 : \quad \mathbb{P}\left(\left|Z^T AZ - \mathbb{E}[Z^T AZ]\right| \geq t\right) \leq C\, \alpha\left(\frac{t}{c\mathbb{E}[\|AZ\|]}\right) + C\, \alpha\left(\sqrt{\frac{t}{c\|A\|}}\right).$$

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t) \qquad (*)$$

with $\alpha : \mathbb{R}^+ \to \mathbb{R}_+$ independent with n.

- $\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty$

- Assume $\|\mathbb{E}[Z]\| \leq \sigma_\alpha$.

Then: $\forall A \in \mathcal{M}_n$, $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\mathbb{P}\left(\left|Z^T AZ - \mathbb{E}[Z^T AZ]\right| \geq t\right) \leq C\ \alpha\left(\frac{ct}{\sigma_\alpha \|A\|_F}\right) + C\alpha\left(\sqrt{\frac{ct}{\|A\|}}\right).$$

$$\begin{aligned}\mathbb{E}[\|AZ\|] &\leq \sqrt{\mathbb{E}[\|AZ\|^2]} \\ &= \sqrt{\mathbb{E}[\mathrm{Tr}(A^T AZZ^T)]} \\ &= \|A\|_F \sqrt{\|\mathbb{E}[ZZ^T]\|}\end{aligned}$$

Lemma: Given $Z \in \mathbb{R}^n$ satisfying $(*)$:

$$\|\mathbb{E}[ZZ^T]\| \leq \|\mathbb{E}[Z]\|^2 + C\sigma_\alpha^2$$

for some numerical constant $C > 0$

IV - Application to Hanson-Wright inequality

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - \mathbb{E}[f(Z)]| \geq t\right) \leq \alpha(t) \qquad (*)$$

with $\alpha : \mathbb{R}^+ \to \mathbb{R}_+$ independent with n.

- $\sigma_\alpha \equiv \sqrt{\int_{\mathbb{R}_+} t\alpha(t)dt} \leq \infty$

- Assume $\|\mathbb{E}[Z]\| \leq \sigma_\alpha$.

Then: $\forall A \in \mathcal{M}_n$, $\forall f : \mathbb{R}^n \to \mathbb{R}$ 1-Lipschitz, $\forall t > 0$:

$$\mathbb{P}\left(\left|Z^T AZ - \mathbb{E}[Z^T AZ]\right| \geq t\right) \leq C\ \alpha\left(\frac{ct}{\sigma_\alpha \|A\|_F}\right) + C\alpha\left(\sqrt{\frac{ct}{\|A\|}}\right).$$

Comparison *Adamczak's result:* $\alpha : t \mapsto e^{-\frac{t^2}{2\sigma_\alpha{}^2}}$

V - Multi-level concentration and conjugate of parallel sum

What happened ?

Compute: $$\left(\frac{\mathrm{Id}}{\nu_1}\right)^{\frac{1}{1}} \boxtimes \min\left(\mathrm{inc}_{\theta_0}, \left(\frac{\mathrm{Id}}{\theta_1}\right)^{\frac{1}{1}}\right)$$

Convention: $$\left(\frac{t}{\theta_0}\right)^{\frac{1}{0}} = \mathrm{inc}_{\theta_0}$$

$$\left(\frac{\mathrm{Id}}{\nu_1}\right)^{\frac{1}{1}} \boxtimes \min\left(\left(\frac{\mathrm{Id}}{\theta_0}\right)^{\frac{1}{0}}, \left(\frac{\mathrm{Id}}{\theta_1}\right)^{\frac{1}{1}}\right) = \min\left(\left(\frac{\mathrm{Id}}{\nu_1\theta_0}\right)^{\frac{1}{1+0}}, \left(\frac{\mathrm{Id}}{\nu_1\theta_1}\right)^{\frac{1}{1+1}}\right).$$

V - Multi-level concentration and conjugate of parallel sum

Theorem:

$$\forall t \in \mathbb{R} : \qquad \alpha(t) = \min_{a \in A} \left(\frac{t}{\check{\alpha}_a} \right)^{\frac{1}{a}} \qquad \text{and} \qquad \beta(t) = \min_{b \in B} \left(\frac{t}{\check{\beta}_b} \right)^{\frac{1}{b}},$$

where $(\check{\alpha}_a)_{a \in A} \in \mathbb{R}_+^A$ and $(\check{\beta}_b)_{b \in B} \in \mathbb{R}_+^B$, for $A, B \subset \mathbb{R}_+$

Then : $$\alpha \boxtimes \beta = \min_{(a,b) \in A \times B} \left(\frac{t}{\check{\alpha}_a \check{\beta}_b} \right)^{\frac{1}{a+b}}.$$

V - Multi-level concentration and conjugate of parallel sum

Theorem: If $\forall t \in \mathbb{R}, \forall k \in [n]$:

$$\alpha^{(k)}(t) = \min_{a \in A^{(k)}} \left(\frac{t}{\sigma_a^{(k)}} \right)^{\frac{1}{a}}$$

where $(\sigma_a^{(k)})_{a \in A^{(k)}} \in \mathbb{R}_+^{A^{(k)}}$ and $A^{(k)} \subset \mathbb{R}_+$

Then : $$\alpha^{(1)} \boxtimes \cdots \boxtimes \alpha^{(n)} = \min_{(a_1, \ldots, a_n) \in A^{(1)} \times \cdots \times A^{(n)}} \left(\frac{t}{\sigma_a^{(1)} \cdots \sigma_b^{(n)}} \right)^{\frac{1}{a_1 + \cdots + a_n}}.$$

Useful when $\|\Phi(Z) - \Phi(Z')\| \leq V_2 \cdots V_n \|Z - Z'\|)$

$\alpha^{(2)}$ $\alpha^{(n)}$ $\alpha^{(1)}$

Proof:

Hypothesis: $\alpha^{(k)}(t) = \min_{a \in A^{(k)}} \left(\frac{t}{\sigma_a^{(k)}} \right)^{\frac{1}{a}}$

$$\left(\alpha^{(1)} \boxtimes \cdots \boxtimes \alpha^{(n)}\right)^{-1} = \left(\inf_{a_1 \in A^{(1)}} \left(\frac{\mathrm{Id}}{\sigma_{a_1}^{(1)}} \right)^{\frac{1}{a_1}} \right)^{-1} \cdots \left(\inf_{a_n \in A^{(n)}} \left(\frac{\mathrm{Id}}{\sigma_{a_n}^{(n)}} \right)^{\frac{1}{a_n}} \right)^{-1}$$

$$= \sup_{a_1 \in A^{(1)}} \sigma_{a_1}^{(1)} \mathrm{Id}^{a_1} \cdots \sup_{a_n \in A^{(n)}} \sigma_{a_n}^{(n)} \mathrm{Id}^{a_n}$$

$$= \sup_{a_1 \in A^{(1)}, \dots, a_n \in A^{(n)}} \sigma_{a_1}^{(1)} \cdots \sigma_{a_n}^{(n)} \mathrm{Id}^{a_1 + \cdots + a_n}$$

$$= \left(\inf_{a_1 \in A^{(1)}, \dots, a_n \in A^{(n)}} \left(\frac{\mathrm{Id}}{\sigma_{a_1}^{(1)} \cdots \sigma_{a_n}^{(n)}} \right)^{\frac{1}{a_1 + \cdots + a_n}} \right)^{-1}$$

V - Multi-level concentration and conjugate of parallel sum

Theorem:

- Consider $Z \in \mathbb{R}^n$, s.t. $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(Z) - f(Z')| \geq t\right) \leq \alpha(t) \qquad (Z, Z' \text{ i.i.d.})$$

- $\Phi : \mathbb{R}^p \to \mathbb{R}^p$: $\qquad \|\Phi(Z) - \Phi(Z')\| \leq \Lambda^{(1)} \cdots \Lambda^{(n)} \cdot \|Z - Z'\|, a.s.$

- $\forall k \in [n]$, $\exists A^{(k)} \subset \mathbb{R}_+$, $(\sigma^{(k)}_{a \in A^{(k)}}) \in \mathbb{R}_+^{A^{(k)}}$:

$$\mathbb{P}\left(\left|\Lambda^{(k)} - \sigma_0^{(k)}\right| \geq t\right) \leq \alpha \circ \inf_{a \in A^{(k)} \setminus \{0\}} \left(\frac{t}{\sigma_a^{(k)}}\right)^{\frac{1}{a}}$$

Then: $\forall f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$\mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z'))| \geq t\right) \leq (n+1)\alpha \circ \inf_{a_k \in A^{(k)}, k \in [n]} \left(\frac{t}{\sigma_{a_1}^{(1)} \cdots \sigma_{a_n}^{(n)}}\right)^{\frac{1}{1 + a_1 + \cdots + a_n}}.$$

Remark:

If $A^{(1)} = \{0\}$: $\Lambda^{(1)} = \sigma^{(1)}$

- Appears in speed
- Do not affect powers

V - Multi-level concentration and conjugate of parallel sum

$$\mathbb{P}\left(|f(\Phi(Z)) - f(\Phi(Z'))| \geq t\right) \leq \alpha \circ \inf_{a_k \in A^{(k)}, k \in [n]} \left(\frac{t}{\sigma^{(1)}_{a_1} \cdots \sigma^{(n)}_{a_n}}\right)^{\frac{1}{1+a_1+\cdots+a_n}}.$$

"Multi-level concentration inequalities"

Our assumption:

$$\|\Phi(Z) - \Phi(Z')\| \leq V_2 \cdots V_n \|Z - Z'\|)$$

Rigorous proofs in:

arXiv > math > arXiv:2402.08206

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Operation with Concentration Inequalities and Conjugate of Parallel Sum

Cosme Louart

THANK YOU!