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Concentration of the Measure Theory to study random matrices

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Classical study of singular values of rectangular RM

$X \in \mathcal{M}_{p,n}$, we study $\frac{1}{n} X X^T$

Classial Hypothesis

- X has i.i.d entries with bounded Variance
- $X = C^{\frac{1}{2}} Z$

Classical conclusions

- Weak convergence of the spectral distribution to the Marcenko-Pastur law

Question : Can we find wider hypothesis and control the speed of convergence ?

With the concentration of the measure theory (CMT)

Hypothesis of CMT

1. For all 1-Lipschitz maps $f : \mathcal{M}_{p,n} \to \mathbb{R}$:

$$\forall t > 0 : \quad \mathbb{P}\left(|f(X) - \mathbb{E}[f(X)]| \geq t\right) \leq 2e^{-t^2/2}$$

2. The column of X are i.i.d.

Remarks

- **(Asset)** True if the columns are Lipschitz transformation of a Gaussian vector $Z \sim \mathcal{N}(0, I_p)$.
  $\longrightarrow$ dependence between the entries of a column possibly complex
- **(Drawback)** That implies that all the moments are bounded

With the concentration of the measure theory (CMT)

Conclusions on the spectral distribution

- Noting $Q(z) = (\frac{1}{n}XX^T + zI_p)^{-1}$, the resolvent of the empirical covariance, $\frac{1}{p}\operatorname{Tr}(Q(z))$ is the *Stieltjes transform* of its spectral distribution and:

$$\forall t > 0: \quad \mathbb{P}\left(\left|\operatorname{Tr}(Q(z)) - \operatorname{Tr}(\tilde{Q}_1)\right| \geq t\right) \leq Ce^{-nt^2/c}, \quad C, c \underset{p,n\to\infty}{=} O($$

where $\tilde{Q}_1 \in \mathcal{M}_p$ is a *deterministic equivalent* of Q

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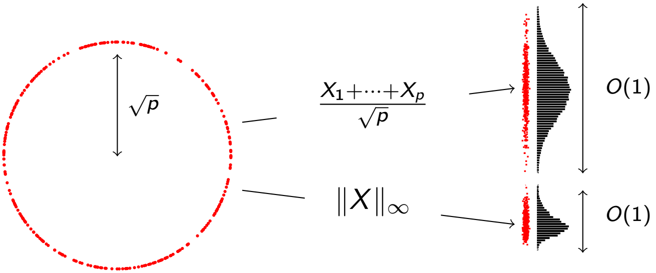
Concentration of the Measure Phenomenon

$$X = (X_1, \ldots, X_p) \sim s_p$$

Observations



Distribution diameter $\underset{p\to\infty}{=} O(\sqrt{p})$

Observable diameter $\underset{p\to\infty}{=} O(1)$

Setting

$(E, \|\cdot\|)$, a normed vector space, $Z \in E$, a random vector

- $(\mathbb{R}^p, \|\cdot\|)$, with $\|x\| = \sqrt{\sum_{i=1}^p x_i^2}$
- $(\mathcal{M}_{p,n}, \|\cdot\|_F)$ with $\|M\|_F = \sqrt{\mathrm{Tr}(MM^T)} = \sqrt{\sum_{\substack{1\leq i\leq p \\ 1\leq j\leq n}} M_{i,j}^2}$

Notations

- if $\exists \alpha : \mathbb{R}^+ \to \mathbb{R}^+ \mid \forall f : E \to \mathbb{R}$ 1-Lipschitz :
  $\boxed{\forall t > 0 \ : \ \mathbb{P}(|f(Z) - \mathbb{E}[f(Z)]| \geq t) \leq \alpha(t),}$ we note $Z \in \alpha$
- In particular, if $\exists \tilde{Z} \in E \mid \forall u : E \to \mathbb{R}$ 1-Lipschitz and linear :
  $\boxed{\forall t > 0 \ : \ \mathbb{P}\left(\left|u(Z - \tilde{Z})\right| \geq t\right) \leq \alpha(t),}$ we note $Z \in \tilde{Z} \pm \alpha$
  $\tilde{Z}$: Deterministic equivalent of Z.
  $(Z \in \alpha \implies Z \in \mathbb{E}[Z] \pm \alpha)$

Standard concentration : Exponential concentration

Fundamental example of the Theory:

$Z \in \mathbb{R}^p$, if Z uniformly distributed on $\sqrt{p}\mathcal{S}^{p-1}$ or $Z \sim \mathcal{N}(0, I_p)$:
$\forall f : E \to \mathbb{R}$ 1-Lipschitz :

$$\forall t > 0 \ : \ \mathbb{P}\left(\left|f(Z) - \mathbb{E}[f(Z')]\right| \geq t\right) \leq 2e^{-t^2/2},$$

For $q, \sigma > 0$, if we note $\mathcal{E}_q(\sigma) : t \mapsto e^{-(t/\sigma)^q}$, then :

$$Z \in 2\mathcal{E}_2(\sqrt{2}) \text{ (Independent of p !).}$$

Standard Hypothesis : $Z \in \tilde{Z} \pm C\mathcal{E}_q(\sigma)$

- $\tilde{Z} \in E$: deterministic equivalent
- $C > 0$, $q > 0$: numerical constants (between $\frac{1}{10}$ and 10)
- $\sigma > 0$: observable diameter, gives the speed of concentration.

How to build new concentrated random vectors ?

- If $Z \in C\mathcal{E}_q(\sigma)$ and $f : E \to E$ λ-Lipschitz, $f(Z) \in C\mathcal{E}_q(\lambda\sigma)$
- No simple way to set the concentration of $(Z_1, \dots, Z_p)$ if $Z_1, \dots, Z_p \in C\mathcal{E}_q(\sigma)$ non independent
- $Z_1, Z_2 \in C\mathcal{E}_q(\sigma)$, independent $(Z_1, Z_2) \in 2C\mathcal{E}_q(2\sigma)$
- $(Z_1, Z_2) = f(Z)$ where $Z \in C\mathcal{E}_q(\sigma)$, and f 1-Lipschitz $(Z_1, Z_2) \in C\mathcal{E}_q(\sigma)$

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Same Notations:

- $\exists a \in \mathbb{R}$ such that:

$$\forall t > 0 \;\; : \;\; \mathbb{P}\left(|Z - a| \geq t\right) \leq Ce^{-(t/\sigma)^q}$$

we note $Z \in a \pm C\mathcal{E}_q(\sigma)$.

Example

$X \sim \mathcal{N}(0, I_p)$, $f : \mathbb{R}^p \to \mathbb{R}$, 1-Lipschitz:

$$f(X) \in \mathbb{E}[f(X)] \pm 2\mathcal{E}_2(\sqrt{2})$$

Characterization with the moments

$$Z \in a \pm C e^{-(\cdot/\sigma)^q}$$

$$\Downarrow ①$$

$$\forall r \geq q :$$
$$\mathbb{E}\left[|Z - a|^r\right] \leq C \left(\frac{r}{q}\right)^{\frac{r}{q}} \sigma^r$$

$$\Downarrow ②$$

$$Z \in a \pm C e^{-\frac{(\cdot/\sigma)^q}{e}}$$

Proof :

1. Fubini:

$$\mathbb{E}\left[|Z-a|^r\right] = \int_Z \left(\int_0^\infty \mathbb{1}_{t\leq |Z-a|^r} dt\right) dZ$$
$$= \int_0^\infty \mathbb{P}\left(|Z-a|^r \geq t\right) dt$$
$$\leq \int_0^\infty Ce^{-t^{\frac{q}{r}}/\sigma^q} dt \ldots \leq C\left(\frac{r}{q}\right)^{\frac{r}{q}} \sigma^r$$

2. Markov inequality:

$$\mathbb{P}\left(|Z-a| \geq t\right) \leq \frac{\mathbb{E}[|Z-a|^r]}{t^r} \leq C\left(\frac{r}{q}\right)^{\frac{r}{q}} \left(\frac{\sigma}{t}\right)^r,$$

with $r = \frac{qt^q}{e\sigma^q} \geq q : \mathbb{P}\left(|Z-a| \geq t\right) \leq Ce^{-(t/\sigma)^q/e}$.

Concentration of the sum

$X \in a \pm C\mathcal{E}_q(\sigma)$, $Y \in b \pm C\mathcal{E}_q(\sigma)$:

- $X + Y \in a + b \pm 2C\mathcal{E}_q\left(2\sigma\right)$
  Proof : $\mathbb{P}\left(|Z_1 + Z_2 - a_1 - a_2| \geq t\right)$

$$\leq \mathbb{P}\left(|Z_1 - a_1| + |Z_2 - a_2| \geq \frac{t}{2} + \frac{t}{2}\right)$$
$$\leq \mathbb{P}\left(|Z_1 - a_1| \geq \frac{t}{2}\right) + \leq \mathbb{P}\left(|Z_2 - a_2| \geq \frac{t}{2}\right)$$
$$\leq 2Ce^{-(t/2\sigma)^q}$$

Concentration of the product

$X \in a \pm C\mathcal{E}_q(\sigma)$ and $Y \in b \pm C\mathcal{E}_q(\sigma)$

- $XY \in ab \pm 2C\mathcal{E}_q\left(3\sigma \max(|a|, |b|)\right) + 2\mathcal{E}_{\frac{q}{2}}\left(3\sigma^2\right)$

Proof : $XY - ab = (X-a)(Y-b) + (X-a)b + (Y-b)a$

$$\begin{aligned}\mathbb{P}\left(|XY - ab| \geq t\right) &\leq \mathbb{P}\left(|X-a| \geq \sqrt{\frac{t}{3}}\right) + \mathbb{P}\left(|Y-b| \geq \sqrt{\frac{t}{3}}\right) \\ &\quad + \mathbb{P}\left(|X-a| \geq \frac{t}{3|b|}\right) + \mathbb{P}\left(|Y-b| \geq \frac{t}{3|a|}\right) \\ &\leq Ce^{-(t/3\sigma^2)^{\frac{q}{2}}} + Ce^{-(t/3|b|\sigma)^q} + Ce^{-(t/3|a|\sigma)^q}\end{aligned}$$

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Control of the norm

- Infinite norm :
$$\mathbb{P}\left(\|Z-\tilde{Z}\|_\infty \geq t\right) = \mathbb{P}\left(\sup_{1\leq i\leq p} e_i^T(Z-\tilde{Z}) \geq t\right)$$
$$\leq p \sup_{1\leq i\leq p} \mathbb{P}\left(e_i^T(Z-\tilde{Z}) \geq t\right) \leq pCe^{-(t/\sigma)^q},$$
- For the general case, use of "ε-nets". If $\exists H \subset (E^*, \|\cdot\|_*)$ |
$$\forall z \in E: \ \|z\| = \sup_{f\in\mathcal{B}_H} f(z).$$

where $\mathcal{B}_H = \{f \in H, \|f\|_* \leq 1\} \subset H$, then :

$$Z \in \tilde{Z} \pm C\mathcal{E}_q(\sigma) \implies \left\|Z-\tilde{Z}\right\| \in 0 \pm 8^{\dim(H)} C\mathcal{E}_q(2\sigma)$$

on $(\mathbb{R}^p, \|\cdot\|)$, $H = \mathbb{R}^p$, and $\dim H = p$

Norm degree

Degree of a subset $H \subset E^*$ and of a norm

- $\eta_H = \log(\# H)$ if H is finite
- $\eta_H = \dim(\mathrm{Vect}(H))$ if H is infinite

Degree of a norm

- $\eta_{\|\cdot\|} = \inf \left\{ \eta_H, H \subset E^* \mid \forall x \in E, \|x\| = \sup_{f \in H} f(x) \right\}$

Example

- $\eta\left(\mathbb{R}^p, \|\cdot\|_\infty\right) = \log(p)$
- $\eta\left(\mathbb{R}^p, \|\cdot\|_r\right) = p$ for $r \geq 1$
- $\eta\left(\mathcal{M}_{p,n}, \|\cdot\|\right) = n + p$
- $\eta\left(\mathcal{M}_{p,n}, \|\cdot\|_F\right) = np.$

Concentration of the norm

If $Z \in \tilde{Z} \pm C\mathcal{E}_q(\sigma)$:

$$\left\| Z - \tilde{Z} \right\| \in 0 \pm C'\mathcal{E}_q(c'\sigma\eta_{\|\cdot\|}^{1/q}) \quad \text{and} \quad \mathbb{E}\left\| Z - \tilde{Z} \right\| \leq C'\sigma\eta_{\|\cdot\|}^{1/q}$$

Example $Z \in \mathbb{R}^p$, $X \in \mathcal{X}_{p,n}$

- if $Z \in \tilde{Z} \pm 2\mathcal{E}_2(\sqrt{2})$: $\mathbb{E}\|Z\| \leq \|\tilde{Z}\| + C\sqrt{p}$
- if $X \in \tilde{X} \pm 2\mathcal{E}_2(\sqrt{2})$: $\mathbb{E}\|X\| \leq \|\tilde{X}\| + C\sqrt{p+n}$,
- if $X \in \tilde{X} \pm 2\mathcal{E}_2(\sqrt{2})$: $\mathbb{E}\|X\| \leq \|\tilde{X}\| + C\sqrt{pn}$.

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Concentration of the sum and the product

If $(X, Y) \in C\mathcal{E}_q(\sigma)$:

- $X + Y \in C\mathcal{E}_q(\sigma)$
- $(X - \tilde{X})(Y - \tilde{Y})$

$$\in C'\mathcal{E}_{\frac{q}{2}}\left(c\sigma^2\right) + C'\mathcal{E}_q\left(c\sigma^2\eta_{\|\cdot\|'}^{\frac{1}{q}}\right) \quad \text{in} \quad (\mathcal{A}, \|\cdot\|)$$

where $\forall x, y \in \mathcal{A}\ \|xy\| \leq \|x\|'\|y\|$ (usually $\|x\|' \leq \|x\|$).

Example $Z \in \mathbb{R}^p$, $X \in \mathcal{M}_{p,n}$, $Z, X \in 2\mathcal{E}_2(\sqrt{2})$

- $\frac{XX^T}{\sqrt{n+p}} \in C\mathcal{E}_2(c) + C\mathcal{E}_1\left(\frac{c}{\sqrt{p+n}}\right)$ in $(\mathcal{M}_{p,n}, \|\cdot\|_F)$
- $\frac{XX^T}{\sqrt{\log(np)}} \in C\mathcal{E}_2(c) + C\mathcal{E}_1\left(\frac{c}{\sqrt{\log(pn)}}\right)$ in $(\mathcal{M}_{p,n}, \|\cdot\|_\infty)$,
- $\frac{Z^{\odot 2}}{\sqrt{\log p}} = \frac{Z \odot Z}{\sqrt{\log p}} \in C\mathcal{E}_2(c) + C\mathcal{E}_1\left(\frac{c}{\sqrt{\log p}}\right)$ in $(\mathbb{R}^p, \|\cdot\|)$

Hanson Wright Theorem

Classical Theorem

If $Z_1, \dots, Z_p \in C\mathcal{E}_2(\sigma)$ independent:

$$\mathbb{P}\left(\left|Z^TAZ - \mathbb{E}Z^TAZ\right| \geq t\right) \leq C \exp\left(-c \min\left(\left(\frac{t}{\sigma^2 \|A\|_F}\right)^2, \frac{t}{\sigma^2 \|A\|}\right)\right)$$

With the Concentration of the measure phenomenon

If $Z = (Z_1, \dots, Z_p) \in C\mathcal{E}_2(\sigma)$:

$$\mathbb{P}\left(\left|Z^TAZ - \mathbb{E}Z^TAZ\right| \geq t\right) \leq C \exp\left(-c \min\left(\left(\frac{t}{\sigma \mathbb{E}\|Z\| \|A\|}\right)^2, \frac{t}{\sigma^2 \|A\|}\right)\right)$$

$\rightarrow$ about the same result since $\mathbb{E}[\|Z\|] \approx \sigma\sqrt{p}$ and $\|A\|_F \approx \sqrt{p}\|A\|$

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Position of the problem

Data matrix $X = (x_1, \ldots, x_n) \in \mathcal{M}_{p,n}$,

Hypothesis:

- $p = O(n)$ and $n = O(p)$
- $X \in C\mathcal{E}_2(c)$
- $\|\mathbb{E}[X]\| = O(\sqrt{n})$

Goal:

Show the concentration of the resolvent:

$$Q = Q(z) = \left(\frac{1}{n} XX^T + zI_p\right)^{-1}$$

and find a computable *deterministic equivalent* $\tilde{Q}_1$ depending on the population covariance : $\Sigma = \frac{1}{n}\mathbb{E}[XX^T]$

Basic results on the resolvent $Q = \left(\frac{1}{n}XX^T + zI_p\right)^{-1}$

- The resolvent is bounded:

$$\|Q(z)\| \leq \frac{1}{z}, \ \left\|Q(z)\frac{XX^T}{n}\right\| \leq 1 \text{ and } \left\|Q(z)\frac{X}{\sqrt{n}}\right\| \leq \frac{1}{z^{1/2}}$$

- $X \mapsto Q(z)$ is $\frac{1}{\sqrt{n}z^{3/2}}$-Lipschitz:

If we note $Q(z)^H = \left(\frac{1}{n}(X+H)(X+H)^T + zI_p\right)^{-1}$:

$$\begin{aligned}
\left\|Q(z)^H - Q(z)\right\|_F &= \left\|\frac{1}{n}Q(z)^H(XX^T - (X+H)(X+H)^T)Q(z)\right\|_F \\
&= \left\|-\frac{1}{n}Q(z)^H HX^T + (X+H)H^T)Q(z)\right\|_F \\
&\leq \frac{1}{\sqrt{n}}\left(\|Q(z)^H\|\left\|\frac{1}{\sqrt{n}}X^TQ\right\| + \left\|\frac{1}{\sqrt{n}}Q^H(X+H)\right\|\|Q(z)\|\right)\|H\|_F
\end{aligned}$$

- $Q(z) \in \mathbb{E}[Q(z)] \pm C\mathcal{E}_2\left(\frac{c}{\sqrt{n}}\right)$ (we suppose that $\frac{1}{z} = O(1)$)

Question

How to estimate $\mathbb{E}\left[\left(\frac{1}{n}XX^T + zI_p\right)^{-1}\right]$ *?*

Design of a Deterministic equivalent

Let $\tilde{\Sigma} \in \mathcal{M}_p$ to be chosen precisely later and we set:

$$\boxed{\tilde{Q}_1 = (\tilde{\Sigma} + zI_p)^{-1}}$$

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With identity $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$:

$$\mathbb{E}[\tilde{Q}_1 - Q] = \mathbb{E}\left[Q\left(\frac{1}{n}XX^T - \tilde{\Sigma}\right)\tilde{Q}_1\right] \quad = \sum_{i=1}^{n}\frac{1}{n}\mathbb{E}\left[Q(x_i x_i^T - \tilde{\Sigma})\tilde{Q}_1\right].$$

Schur formulas

We set $X_{-i} = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in \mathcal{M}_{p,n}$
and $Q_{-i} = (\frac{1}{n}X_{-i}X_{-i}^T + zI_p)^{-1}$:

$$Q = Q_{-i} - \frac{1}{n}\frac{Q_{-i}x_i x_i^T Q_{-i}}{1 + \frac{1}{n}x_i^T Q_{-i}x_i} \quad \text{and} \quad Qx_i = \frac{Q_{-i}x_i}{1 + \frac{1}{n}x_i^T Q_{-i}x_i}.$$

Then:

$$\tilde{Q}_1 - \mathbb{E}Q = \sum_{i=1}^{n}\frac{1}{n}\mathbb{E}\left[Q_{-i}\left(\frac{x_i x_i^T}{1 + \frac{1}{n}x_i^T Q_{-i}x_i} - \tilde{\Sigma}\right)\tilde{Q}_1\right] - \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}\left[Q_{-i}x_i x_i^T Q\tilde{\Sigma}\tilde{Q}_1\right].$$

A first deterministic equivalent

$$\left\| \tilde{Q}_1 - \mathbb{E}Q \right\| = \sup_{\|u\|,\|v\|\leq 1} u^T \left(\tilde{Q}_1 - \mathbb{E}Q \right) v$$
$$= \sup_{\|u\|,\|v\|\leq 1} \frac{1}{n} \sum_{i=1}^{n} \Delta_i + \varepsilon_i$$

with:

- $\Delta_i = \mathbb{E}\left[u^T Q_{-i} \left(\frac{x_i x_i^T}{1+\frac{1}{n} x_i^T Q_{-i} x_i} - \tilde{\Sigma} \right) \tilde{Q}_1 v \right]$
- $\varepsilon_i = \frac{1}{n}\mathbb{E}\left[u^T Q_{-i} x_i x_i^T Q \tilde{\Sigma} \tilde{Q}_1 v \right]$

$\longrightarrow$ we note $\delta_1 = \frac{1}{n}\mathrm{Tr}(\Sigma \mathbb{E}[Q])$ and we chose $\boxed{\tilde{\Sigma} = \frac{\Sigma}{1+\delta_1}}$

Let us show that with this choice: $\Delta_i, \varepsilon_i = O\left(\frac{1}{\sqrt{n}}\right)$

Preliminary lemmas

- $u^T Q x_i = \frac{1}{\sqrt{n}}(\sqrt{n} u^T Q) x_i \in C\mathcal{E}_2(c) + C\mathcal{E}_1\left(\frac{c}{\sqrt{p}}\right)$
- $$\mathbb{E}[u^T Q x_i] \leq \sqrt{\mathbb{E}[u^T Q x_i x_i^T Q u]} = \sqrt{\frac{1}{n}\mathbb{E}[u^T Q X X^T Q u]}.$$
  $$\leq \mathbb{E}[u^T Q u] = O(1)$$
- The same way:
  $u^T Q_{-i} x_i, u^T \tilde{Q}_1 x_i \in O(1) \pm C\mathcal{E}_2(c) + C\mathcal{E}_1\left(\frac{c}{\sqrt{p}}\right)$

Preliminary lemmas

- $\frac{1}{n} x_i^T Q_{-i} x_i \in C\mathcal{E}_2(c) + C\mathcal{E}_1\left(\frac{c}{\sqrt{n}}\right)$
- $\mathbb{E}\left[\frac{1}{n} x_i^T Q_{-i} x_i\right] = \frac{1}{n}\mathrm{Tr}(\Sigma \mathbb{E}[Q_{-i}]) \leq \frac{1}{n}\mathrm{Tr}(\Sigma)\mathbb{E}\left[\|Q_{-i}\|\right] = O(1)$
- $$\begin{aligned} \|\mathbb{E}Q_{-i} - \mathbb{E}Q\| &= \sup_{\|u\|,\|v\|\leq 1} u^T \left(\mathbb{E}Q_{-i} - \mathbb{E}Q\right) v \\ &= \sup_{\|u\|,\|v\|\leq 1} \mathbb{E}\left[\frac{1}{n} u^T Q_{-i} x_i x_i^T Q v\right] = O\left(\frac{1}{n}\right) \end{aligned}$$
- $\frac{1}{n} x_i^T Q_{-i} x_i \in \delta_1 \pm C\mathcal{E}_2(c) + C\mathcal{E}_1\left(\frac{c}{\sqrt{n}}\right)$ (recall that $\delta_1 = \frac{1}{n}\mathrm{Tr}(\Sigma \mathbb{E}[Q]))$

End of the proof of the estimation with the first deterministic equivalent $\tilde{Q}_1 = \left(\frac{\Sigma}{1+\delta_1} + zI_p\right)^{-1}$

- Since $\left\|\tilde{\Sigma}\tilde{Q}_1\right\| = O(1)$, $\varepsilon_i = \frac{1}{n}\mathbb{E}\left[u^T Q_{-i} x_i x_i^T Q \tilde{\Sigma}\tilde{Q}_1 v\right] = O\left(\frac{1}{n}\right)$
- $\Delta_i = \mathbb{E}\left[u^T Q_{-i}\left(\frac{x_i x_i^T}{1+\frac{1}{n}x_i^T Q_{-i} x_i} - \frac{\Sigma}{1+\delta_1}\right)\tilde{Q}_1 v\right]$

$$= \mathbb{E}\left[\frac{u^T Q_{-i} x_i x_i^T \tilde{Q}_1 v(\delta_1 - \frac{1}{n}x_i^T Q_{-i} x_i)}{\left(1+\frac{1}{n}x_i^T Q_{-i}x_i\right)(1+\delta_1)}\right]$$

$$+ \mathbb{E}\left[u^T Q_{-i}\left(\frac{x_i x_i^T - \Sigma}{1+\delta_1}\right)\tilde{Q}_1 v\right]$$

$$= O\left(\frac{1}{\sqrt{n}}\right)$$

$$\Longrightarrow \left\|\mathbb{E}[Q] - \tilde{Q}_1\right\| = O\left(\frac{1}{\sqrt{n}}\right)$$

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Second deterministic equivalent

Note that $\delta_1 = \frac{1}{n}\mathrm{Tr}(\Sigma\mathbb{E}[Q]) = \frac{1}{n}\mathrm{Tr}(\Sigma\tilde{Q}_1) + O\left(\frac{1}{\sqrt{n}}\right)$

$$= \frac{1}{n}\mathrm{Tr}\left(\Sigma\left(\frac{\Sigma}{1+\delta_1} + zI_p\right)^{-1}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

The function

$$\begin{aligned} \mathbb{R}^+ &\longrightarrow \quad \mathbb{R}^+ \\ \delta &\longmapsto \quad \frac{1}{n}\mathrm{Tr}\left(\Sigma\left(\frac{\Sigma}{1+\delta} + zI_p\right)^{-1}\right) \end{aligned}$$

is contracting for the semimetric: $d_s(\delta, \delta') = \frac{|\delta-\delta'|}{\sqrt{\delta\delta'}}$

$\Longrightarrow$ It admits a unique fixed point:

$$\delta_2 = \frac{1}{n}\mathrm{Tr}\left(\Sigma\left(\frac{\Sigma}{1+\delta_2} + zI_p\right)^{-1}\right)$$

End of the proof

It can be showed that $\delta_1 - \delta_2 = O\left(\frac{1}{\sqrt{n}}\right)$ thus if we set

$$\boxed{\tilde{Q}_2 = \left(\frac{\Sigma}{1+\delta_2} + zI_p\right)^{-1}}:$$

$$\left\|\mathbb{E}[Q] - \tilde{Q}_2\right\| \leq \left\|\mathbb{E}[Q] - \tilde{Q}_1\right\| + \left\|\tilde{Q}_1 - \tilde{Q}_2\right\|$$
$$\leq \left\|\tilde{Q}_1 \frac{\Sigma(\delta_2 - \delta_1)}{(1+\delta_2)(1+\delta_1)} \tilde{Q}_2\right\| + O\left(\frac{1}{\sqrt{n}}\right) \quad = \quad O\left(\frac{1}{\sqrt{n}}\right)$$

$$\boxed{\Longrightarrow \forall t > 0 \;:\; \mathbb{P}\left(\left|\frac{1}{p}\operatorname{Tr}(Q) - \frac{1}{p}\operatorname{Tr}(\tilde{Q}_2)\right| \geq t\right) \leq Ce^{-cnt^2}}$$